Government College of Engineering and Research
Avasari, Pune

Fundamental of Finite Element Analysis

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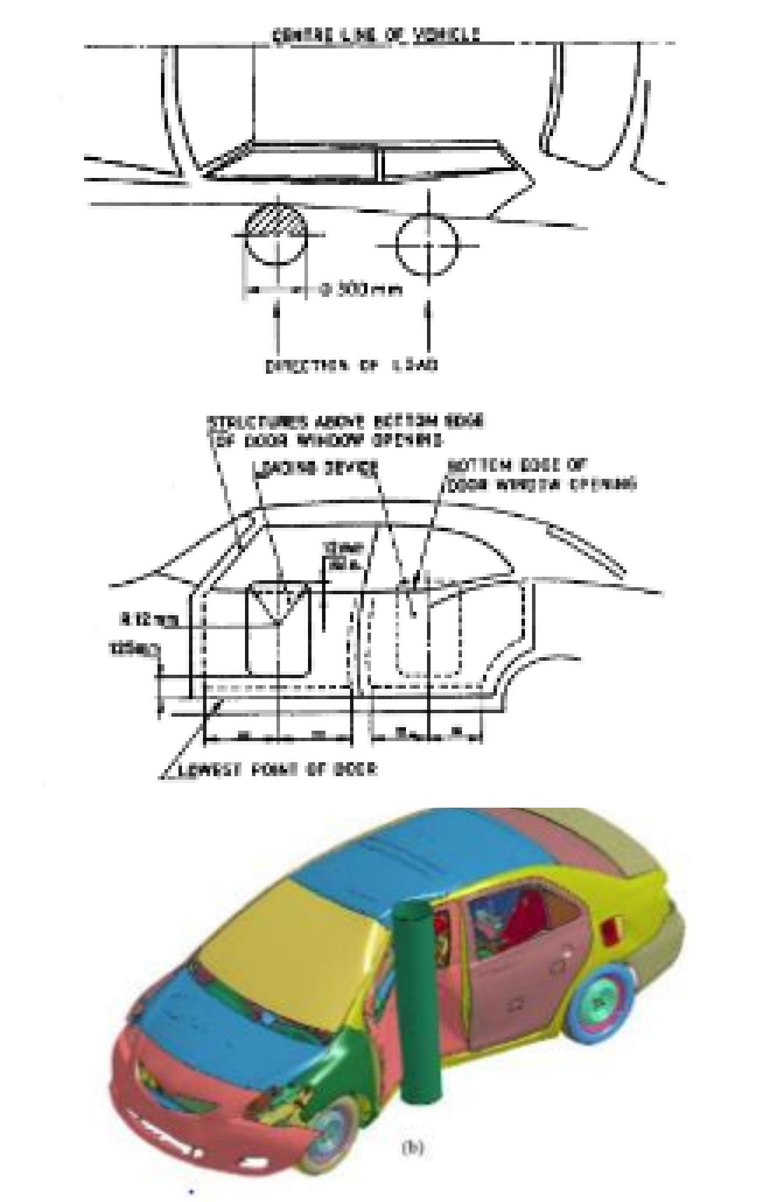
Course Outcomes

On completion of the course, learner will be able to,

- Analyzed 1-D and 2-D solid mechanics problem by using FEM
- Analyzed 1-D heat transfer problem by using FEM
- Analyzed 1-D modal analysis problem by using FEM
- Explain the inner workings of a finite element code for linear stress, displacement, temperature and modal analysis
- Interpret the results of finite element analyses and make an assessment of the results in terms of modeling (physics assumptions) errors, discretization (mesh density and refinement toward convergence) errors, and numerical (round off) errors.

Course Motivation

IS 12009: Automotive Vehicle - Safety Requirements for Side Door of Passenger

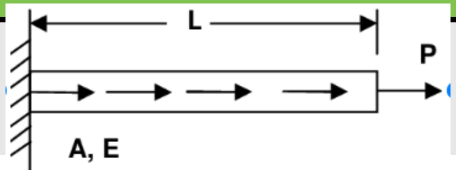


As per the Indian standards IS 12009,

- A long cylindrical or semi cylindrical rigid steel bar of 300mm diameter is intruded along the circumferential surface on the side door from outside.
- The initial crush resistance, Intermediate crush resistance and peak crush
  resistances are measured at 150mm, 300mm and 450mm of crush distance respectively.
- As per the standards, the initial crush resistance should not be less than 10KN or 0.6 times of kerb weight of vehicle.
- The Intermediate crush resistance should not be less than 20KN or 1.2 times of kerb weight of vehicle
- The peak crush resistances should not be less than 55kN or 2 times of kerb weight of vehicle

Unit 1
Fundamentals concepts of FEA

Method of Solving the Engineering Problem

Analytical Method	Numerical Method	Experimental Method
		
δl= PL/AE	[F]= [K]{u}	Experimentally performed
Classical Approach	Mathematical (Numerical) Approach	Actual Measurement
High degree of accurate result	Approximate result	Gives actual result but accuracy depends on experimental setup and performance
Applicable for Simple (standard shape) problem	Real life Problem	Prototype need for experiment
No/less costly as it is analytical approach	Cost of software are include in product cost	Cost depends on experimental setup/ prototype requirement/ time on such factor

Finite Element Method

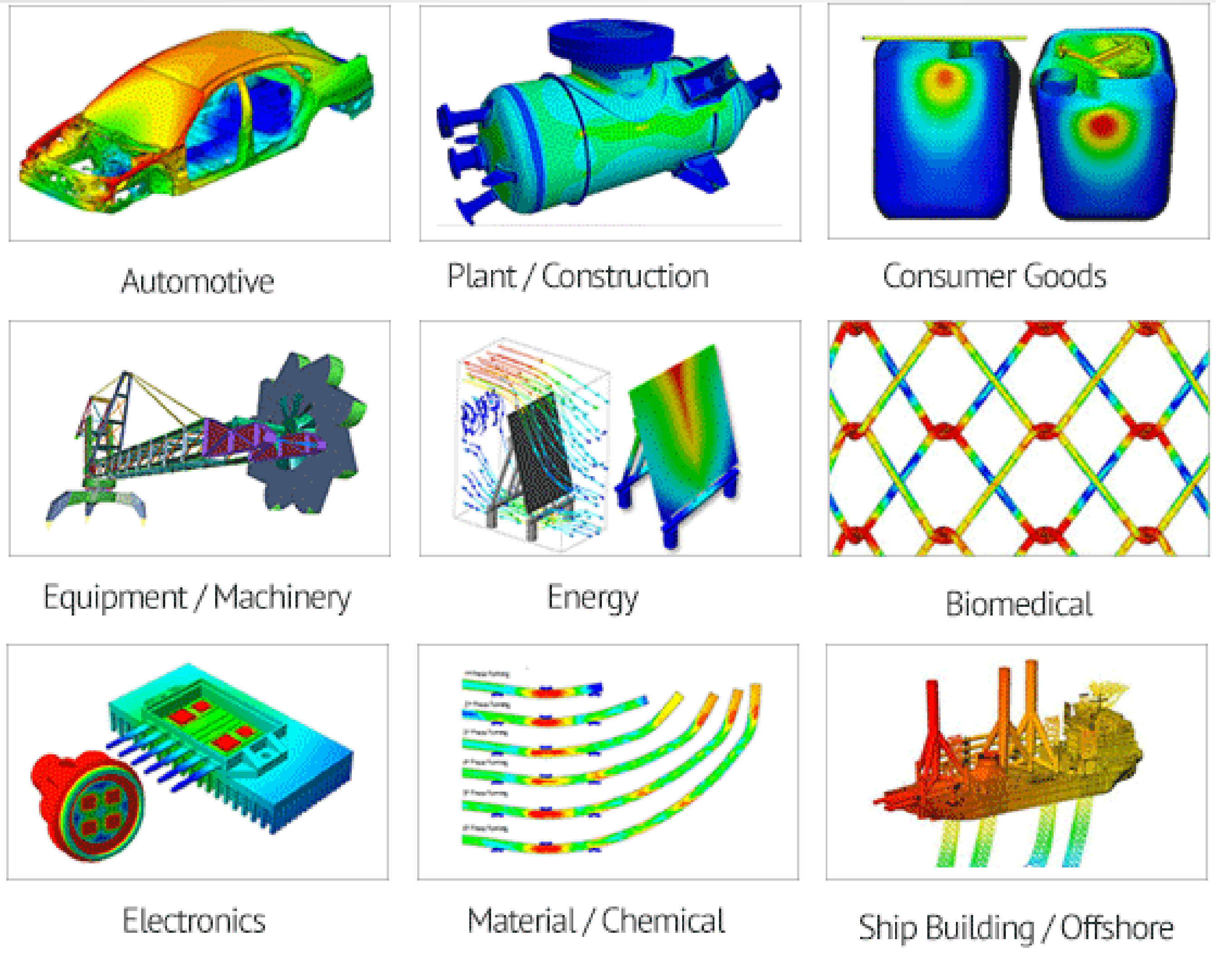
- The FEM is a computer aided mathematical technique that is used to obtain an approximate numerical solution to the fundamental differential and/or integral equation that predict the response of physical system to external effect

- Use for problem with complicated geometries, loading, material properties where analytical solution can not be obtained

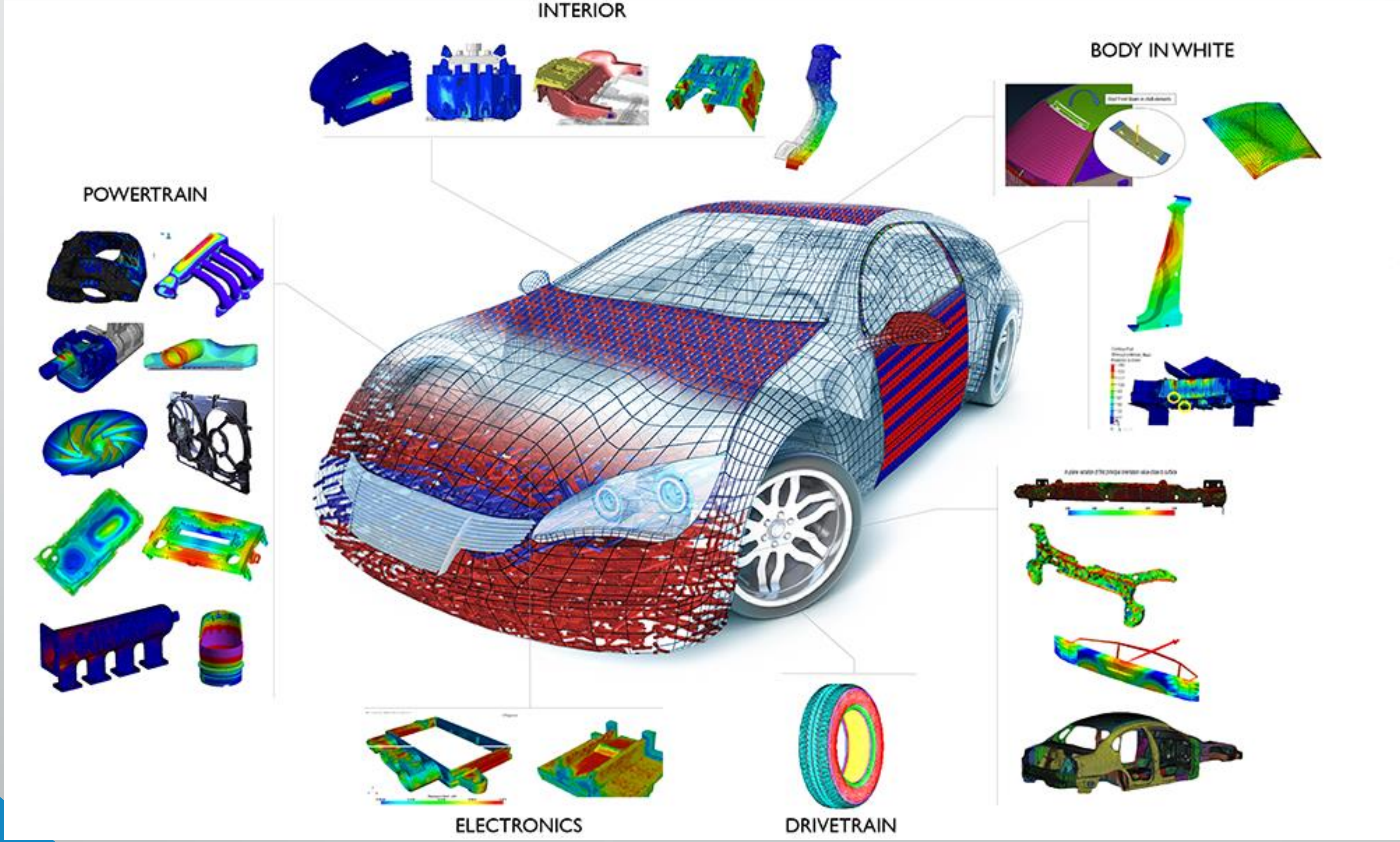
Brief History of Finite Element Method

- 1909 ---- Ritz Introduce the variation method is
- 1915---- Galerikin introduce the Weighted residual method
- 1940--- Basic ideas of FEA were developed by aircraft engineer, they use matrix method
- 1943----- Courant R. used Variational methods for the solution of problems of equilibrium and vibrations
- 1945---- Hrenni koff used in structural field
- 1947 ----- Levy introduce flexibility/ force method
- 1953----- Argyris & Kelsey used matrix for structural analysis
- 1960----- Dr. Ray Clough, coined the term “finite element” in The this times saw the true beginning of commercial FEA as digital computers replaced analog ones with the capability of thousands of operations per second.
- 1961---- Turner used for large deflection and thermal problem
- 1962---- Zinkiewicz– visco elasticity problem
- 1990---- Analysis of large structural problem

Applications of FEM in various field



Applications of FEM in Automobile



Advantage of FEM

- Analysis problem contain the Model irregularly shaped bodies
- Handle general/real life load conditions
- Handle the variation of material within the component quite easily
- Solve problem with various boundary conditions
- Alter finite element model relative easy and cheap and time saver
- Handle nonlinear behaviour in material boundary etc.

Limitation of FEM

- FEM is applied to an approximation of the mathematical model of system
- Experience and judgment needed in order to construct the good finite element model
- A powerful computer and reliable FEM software are essential
- Accuracy of obtain solution is usually a function of the mesh resolution
- Numerical error such as the limitation of number of significant digits, rounding off occur very often
- Many time result of FEM need to be verify by various methods

Numerical method for Engineering Problem

- Finite Element Method
- Finite Differential Method
- Finite Volume method
- Boundary value method

Similarities that exists between various types of engineering problems:

1. Solid Bar under Axial Load

$$\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)=Q,------GE$$

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right)=0,$$

Where , E is the Young's modulus,

u is axial displacement,

and A is cross – sectional area

2. One – dimensional Heat Transfer

$$\frac{\partial}{\partial x}\left(KA\frac{\partial T}{\partial x}\right)=0, \text{Laplace equation}$$

Where , K is the thermal conductivity,

T is temperature, and A is cross – sectional area

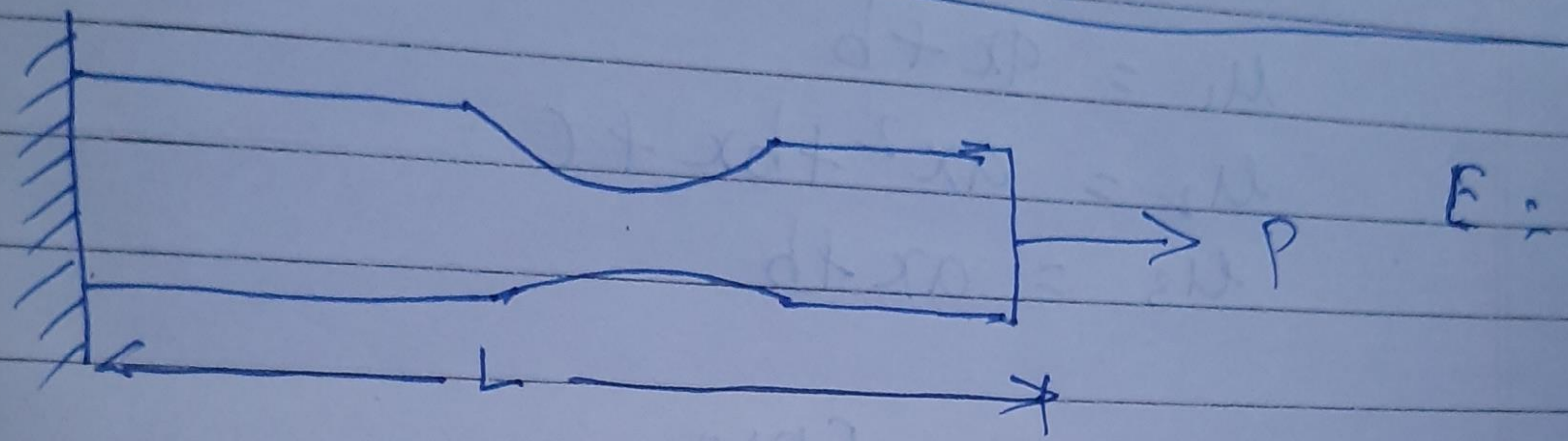
3. One dimensional fluid flow

$$\frac{\partial}{\partial x}\left(\rho A\frac{\partial \Phi}{\partial x}\right)=0, \text{Where ,}$$

ρ is the density, φ = is potential function ,

and A is cross – sectional area and $u=\frac{\partial \Phi}{\partial x}$

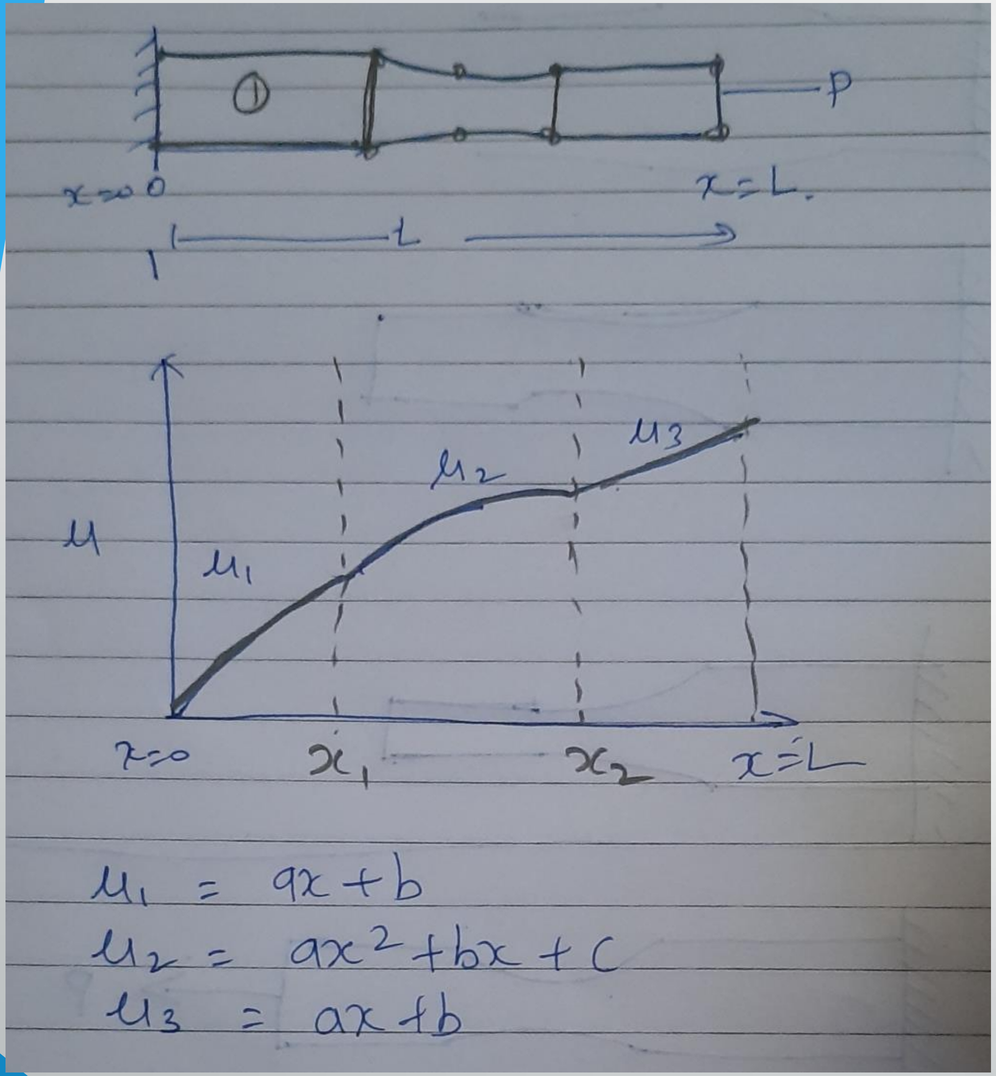
FEM and FDM



$$-\frac{d}{dx}\left[EA\frac{du}{dx}\right]=0$$

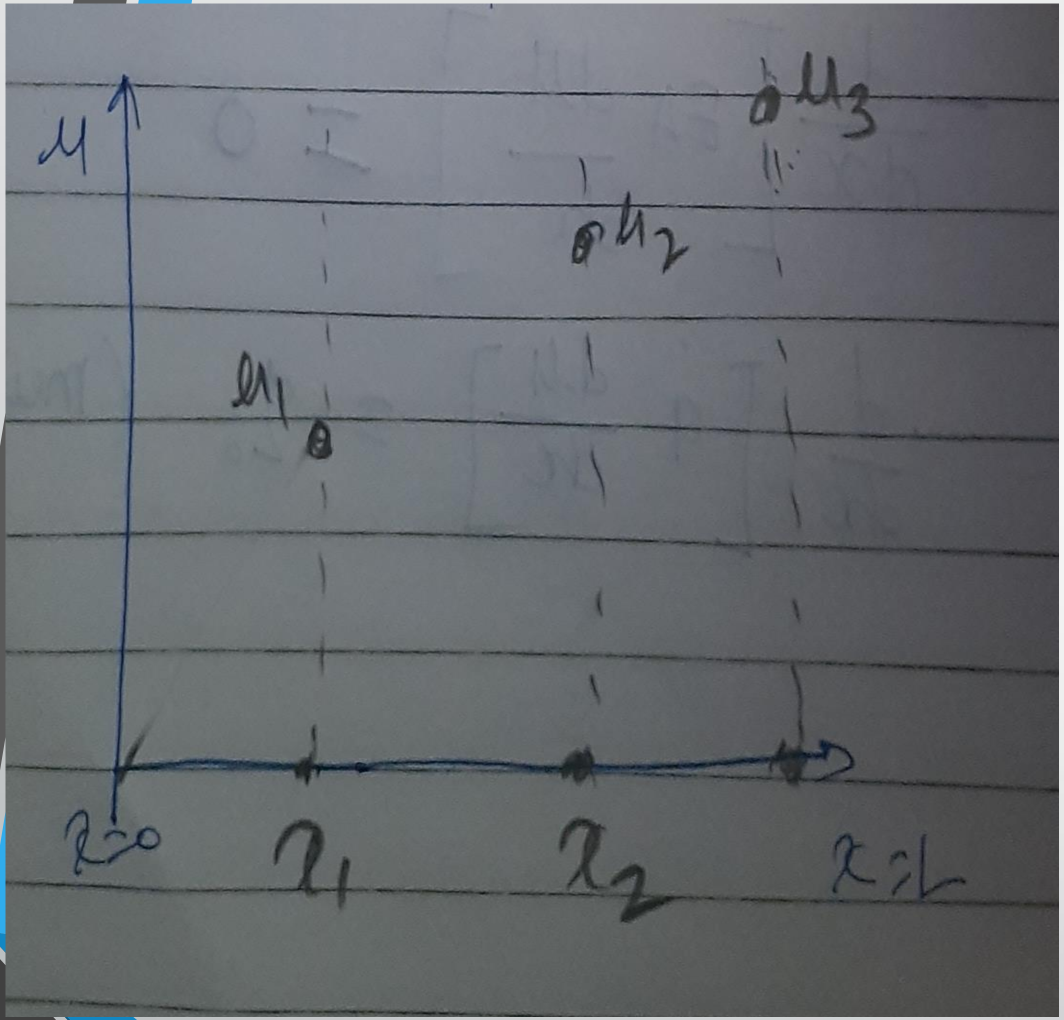
$$-\frac{d}{dx}\left[a\frac{du}{dx}\right]=q_0 \quad \text{(main G.E.)}$$

FEM



- Formulate equations for each element.
- Approximating the fields variable within each element represent by a simple function, such as a linear or quadratic polynomial, with a finite number of degrees of freedom (DOFs).

FDM



- The finite-difference method is the most direct approach to discretizing differential equations
- Field variables are calculated at a point, hence for irregular shape accuracy of results are comparatively less

Features Finite Element Method

- The finite-element method is a computational method that subdivides a physical geometry into very small but finite-sized elements of geometrically simple shapes
- Formulate equations for each element.
- Approximating the fields variable within each element represent by a simple function, such as a linear or quadratic polynomial, with a finite number of degrees of freedom (DOFs).
- Contributions from all elements are assembled you end up with a large sparse matrix equation
- finite-element method the most difficult
- In FEM it easy to "increase the order of the elements" so that the physics fields can be approximated very accurately
- In multi-physics analysis it is used effectively
- Curved and irregular geometries are handled effectively

Features Finite Difference Method

- The finite-difference method is the most direct approach to discretizing differential equations
- Field variables are calculated at a point, hence for irregular shape accuracy of results are comparatively less
- Formulate equations for each element
- Required less computational power
- The finite-difference method is more difficult to use for handling material discontinuities
- The finite-difference method is difficult to use for handling multi-physics problems

Approximate Methods of solving differential equations

- Weighted residual method
  a. Galerkin method
  b. Collection method
  c. Subdomain method
  d. Least square method
- Variational method
  a. Rayleigh Ritz method

Weighted residual method

Let us consider the following differential equation:

$$\mathbb{B}(u(x)) = \frac{d^2u(x)}{dx^2} + u(x) \quad \text{on} \quad \Omega = [0, 1]$$

with boundary conditions

$$u(x = 0) = 1$$
$$u(x = 1) = 0$$

This differential equation has an exact solution given by

$$u(x) = 1 - \frac{\sin(x)}{\sin(1)}$$

- Let us solve the differential equation using the weighted residual method . We choose the approximating function u(x) in the form of a polynomial: $u(x) = 1 + \alpha_1 x + \alpha_2 x^2$

- To ensure that the trial function *u(x)* approximate the exact function *u(x)* as best as possible, we need to make sure that it is derivable as many times as required by the differential operator and satisfies the boundary conditions; that is,

$$u(x) = 1 + \alpha_1 x + \alpha_2 x^2 \quad \Rightarrow \quad u(x = 0) = 1$$

$$u(x = 1) = 0 \Rightarrow 1 + \alpha_1 + \alpha_2 = 0$$
$$\Rightarrow \alpha_1 = -(1 + \alpha_2)$$

The trial function therefore becomes

$$u(x) = \alpha_2(x^2 - x) - x + 1$$

$$\mathbb{R}(\overline{u}(x)) = \frac{d^2\overline{u}(x)}{dx^2} + \overline{u}(x)$$

By putting approximate function in given equation

Residual

$$= \alpha_2 (x^2 - x + 2) - x$$

To minimize residual over the domain following method are used,

a. Galerkin method
b. Collection method
c. Subdomain method
d. Least square method

a. Galerkin Method

In Galerkin method,

- weighted function is choose
- multiply w.f. to residual and
- integrate in entire domain

The corresponding weighting function is obtained as

$$\psi = \delta\overline{u}(x) = \delta\alpha_2(x^2 - x)$$

Integrating the product of the weighted residual over the domain yields

$$W = \int_0^{+1} \delta\alpha_2(x^2 - x) \times (\alpha_2(x^2 - x + 2) - x)\, dx = 0$$

Since $\delta\alpha_2 \neq 0$, it follows

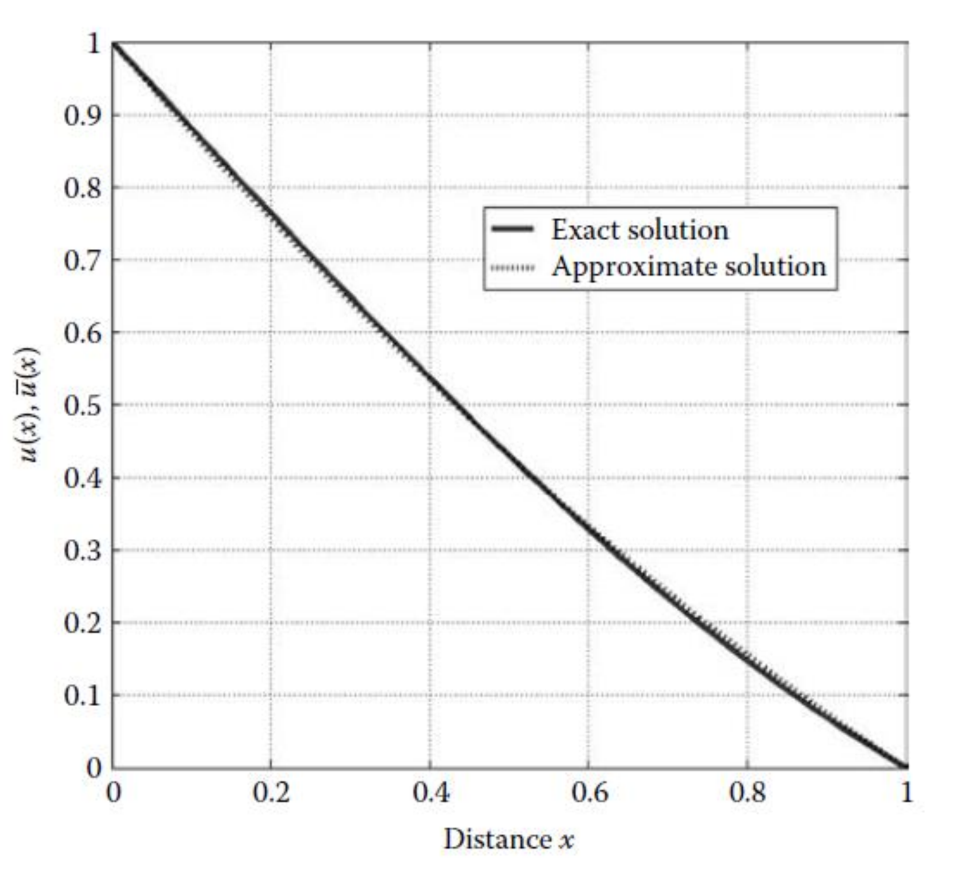
$$W = \int_0^{+1} (x^2 - x) \times (\alpha_2(x^2 - x + 2) - x)\, dx = 0$$

Evaluating the integral leads to an algebraic equation of the form

$$\frac{1}{12} - \frac{3}{10}\alpha_2 = 0 \Rightarrow \alpha_2 = \frac{5}{18}$$

The final approximation is then written as

$$\overline{u}(x) = \frac{5}{18}(x^2 - x) - x + 1$$



u(x) = 0.277(x2 − x) − x + 1

b. Collection method

In this method the residual is set equal to zero at *n* distinct points in the solution domain to value to α_2

$$R(x_j) = 0$$

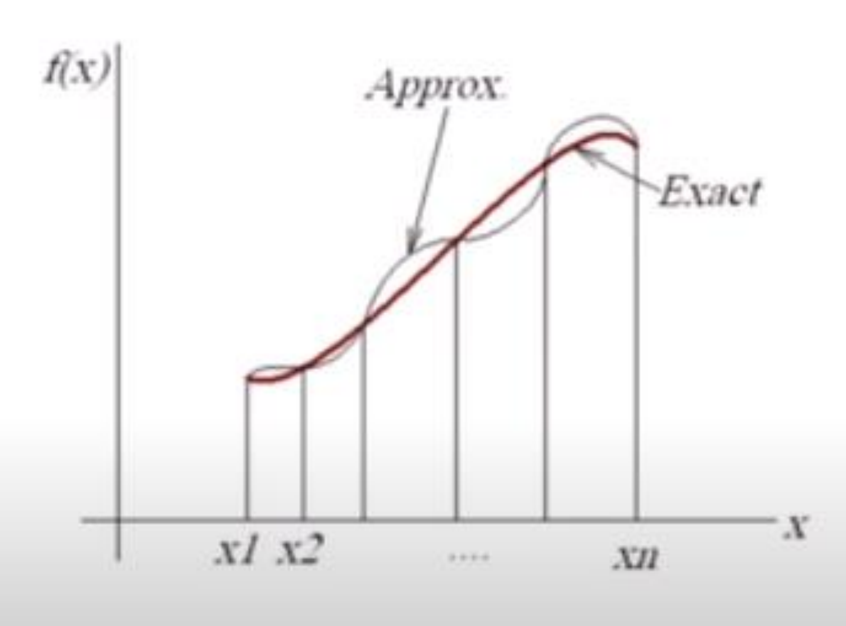
Residual $= \alpha_2(x^2 - x + 2) - x$

Residual at x=0.5

$= \alpha 2\ (0.5^2 + 0.5 + 2)\ -0.5$

$\alpha 2 = 0.2857$

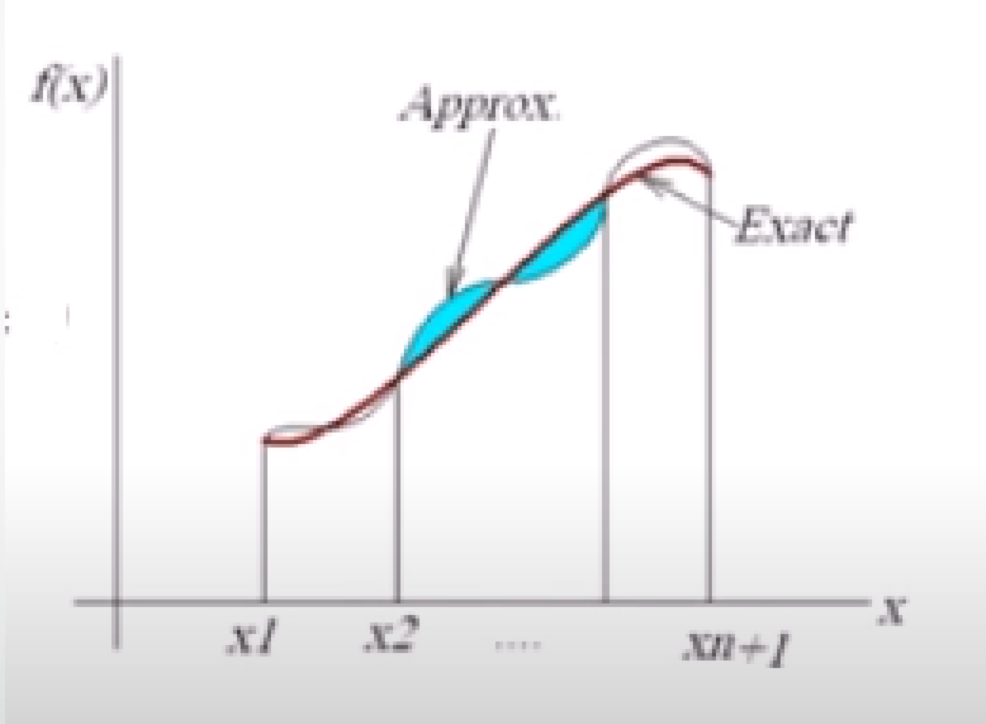
$u(x) = 0.2857(x2 - x) - x + 1$



Subdomain method

The idea behind the subdomain method is to force the integral of residue to be equal to zero on subinterval of domain.

No any weight function considered



Rayleigh –Ritz (Variation) Method

Let u be the displacement of the spring under the load P. We then have

$$\mathcal{U} = \tfrac{1}{2}ku^2; \qquad \mathcal{W} = Pu$$

and

$$\Pi = \tfrac{1}{2}ku^2 - Pu$$

Note that for a given P, we could graph Π as a function of u. Using (3.1) we have, with u as the only variable,

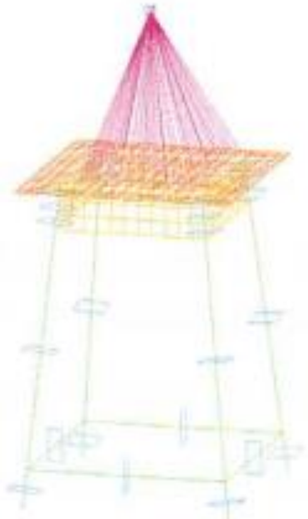
$$\delta\Pi = (ku - P)\,\delta u; \qquad \frac{\partial\Pi}{\partial u} = ku - P$$

which gives the equilibrium equation

$$ku = P \tag{a}$$

Using (a) to evaluate $\mathcal{W}$, we have *at equilibrium* $\mathcal{W} = ku^2$; i.e., $\mathcal{W} = 2\mathcal{U}$ and $\Pi = -\tfrac{1}{2}ku^2 = -\tfrac{1}{2}Pu$. Also, $\partial^2\Pi/\partial u^2 = k$ and hence at equilibrium Π is at its minimum.

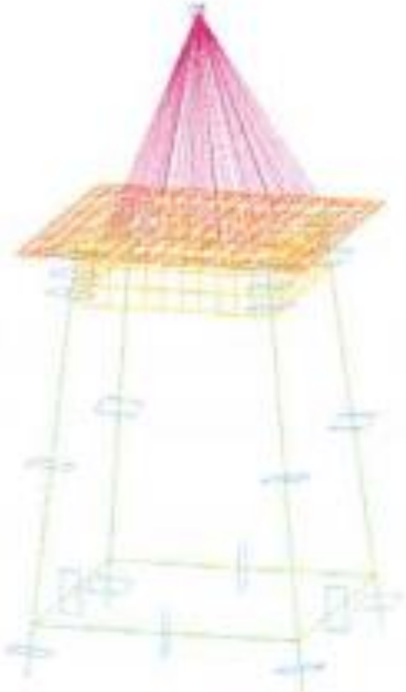
How the Finite Element Method Works



- **Discretize the continuum:** Divide the continuum or solution region into elements.
- **Select interpolation functions:** Assign nodes to each element and then choose the interpolation function to represent the variation of the field variable over the element.
- **Find the Element Properties:** Determine the matrix equations expressing the properties of the individual elements. For this one of the three approaches can be used. i) The direct approach ii) The variational approach or iii) the weighted residuals approach.

- **Assemble the Element Properties to Obtain the System equations:** Combine the matrix equations expressing the behavior of the elements and form the matrix equations expressing the behavior of the entire system.
- **Impose the Boundary Conditions:** Before the system equations are ready for solution they must be modified to account for the boundary conditions of the problem.
- **Solve the System Equations:** Solve the System Equations to obtain the unknown nodal values like displacement, temperature etc.
- **Make Additional Computations If Desired:** From displacements calculate element strains and stresses, from temperatures calculate heat fluxes if required.

How the Finite Element Software work



Real life problem

Step 1: Pre Processing

A) CAD data

B) Meshing

C) Boundary Condition

Step 2) Processing and Solution

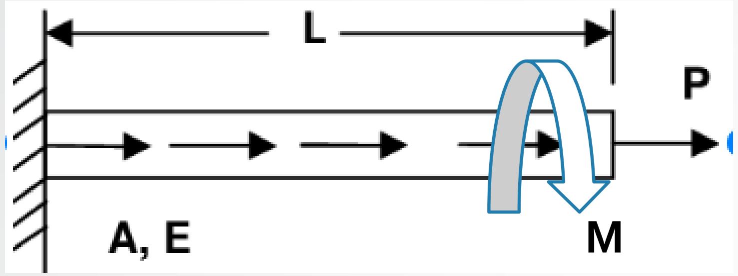
Step 3) Post Processing

CAD model

Meshing and Boundary Condition

Post Processing

Type of boundary condition



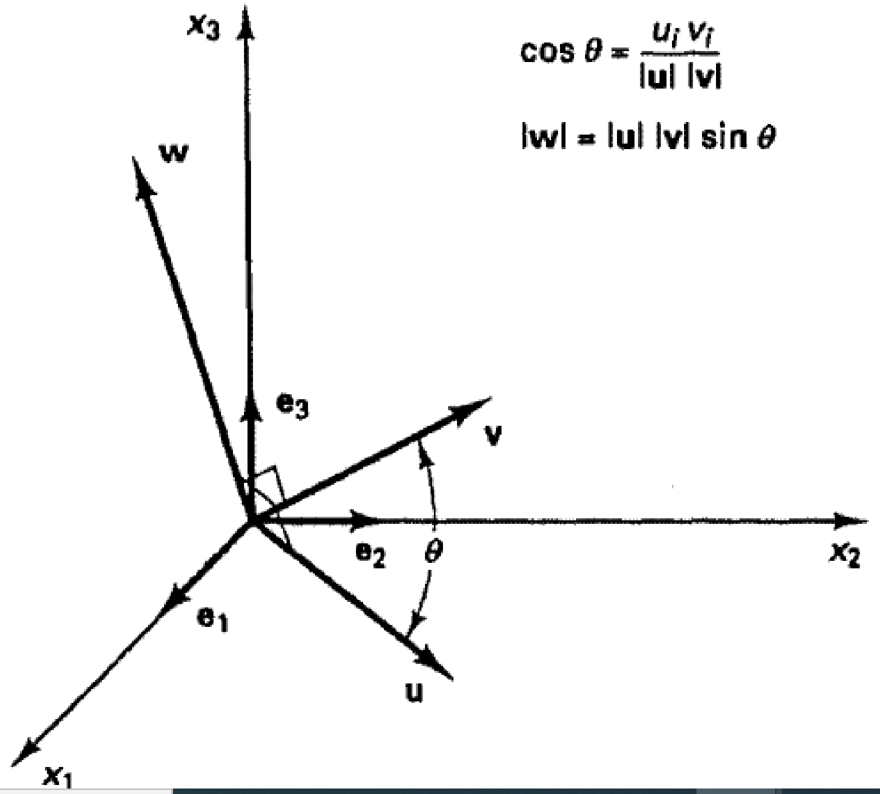
They are two classes of boundary conditions, called as

- ***essential boundary conditions***
- ***natural boundary conditions.***

Essential Boundary Conditions : The essential boundary conditions are also called *geometric boundary conditions* in the structural mechanics the essential boundary conditions correspond to prescribed displacements and rotations

Natural Boundary Conditions. This natural boundary conditions, are also called *force boundary conditions* because in structural mechanics the natural boundary conditions correspond to prescribed boundary forces and moments

Vector



$$\cos\theta = \frac{u_i v_i}{|u|\,|v|}$$

$$|w| = |u|\,|v| \sin\theta$$

$$\mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}; \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Let us consider the following differential equation:

$$\mathbb{B}(u(x)) = \frac{d^2u(x)}{dx^2} + u(x) \text{ on } \Omega = [0, 1]$$

with boundary conditions

$$u(x = 0) = 1$$
$$u(x = 1) = 0$$

$$|\mathbf{u}| = 3\sqrt{2}$$
$$|\mathbf{v}| = 2\sqrt{2}$$

Hence

$$\cos\theta = \tfrac{1}{2}$$

and $\theta = 60°$.

A vector perpendicular to the plane defined by **u** and **v** is given by

$$\mathbf{w} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

hence

$$\mathbf{w} = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix}$$

Using $|\mathbf{w}| = \sqrt{w_i w_i}$, we obtain

$$|\mathbf{w}| = 6\sqrt{3}$$

which is also equal to the value obtained using the formula given in Fig. 2.4.

Thank You
For Your Attention